

A Historical Persp

Using quadratic functions as an example, see how the evolution of technology has altered the expectations of students' understanding and critical thinking.

Evolving technology has played an important part in a common quadratic-function lesson. Having been mentioned repeatedly in numerous reform documents, a recurring lesson has involved changing the parameters in $f(x) = ax^2 + bx + c$ and studying the effects on the graph.

In both NCTM Yearbooks and NCTM Standards documents, technology is discussed and explored as a way to enrich quadratic function tasks. However, we will show how advances in technology have enriched these activities.

A TI-73 Explorer calculator is shown in the upper right quadrant of the page. The calculator is blue and black, with a monochrome LCD screen displaying several rows of numbers and the text 'L1:GREEN'. The background of the entire page is a blue gradient with a pattern of binary code (0s and 1s) that appears to be receding into the distance, creating a sense of depth and technology.

ective from *a* to *c*

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THE APPROACH BEFORE TECHNOLOGY

In the 1959 NCTM Yearbook, Smith and Henderson discussed the importance of inductive reasoning and making conjectures while doing proofs. They asked students to study the effect on the graph of changing each coefficient while holding the other two coefficients constant (see **fig. 1a**). In 1959, these graphs had to be produced by hand. Smith and Henderson also asked students to study the roots of different quadratic functions

graphically and collect data from their graphs to create a table to study the relationship between the product/sum of the roots and the coefficients *a*, *b*, and *c* (see **fig. 1b**).

In a state-of-the-art classroom in 1959, students would have created graphs by hand on graph paper, similar to those in **figure 2**. Graphing by hand is a time-consuming, tedious process. To generate enough graphs to enable students to make conjectures about the effects of *a*, *b*, and *c*, the teacher might have asked each student

or group of students (perhaps by row) to draw just one graph in each family. In studying the effect of *c*, for example, the students in each of five different rows might have been assigned to graph $y = x^2$; $y = x^2 + 2$; $y = x^2 + 4$; $y = x^2 - 2$; or $y = x^2 - 4$.

Once all the graphs were completed, students might have compared their graphs. Then the teacher might have guided them to the conclusion that changing *c* causes a vertical translation of the graph. The effects of *a* and *b* could be studied in a similar

fashion. However, the time investment necessary to complete the graphs by hand would have made investigating the effects of all three coefficients during one class period very difficult. Also, the challenge of generating enough graphs by hand during a single class may have rendered such investigations impractical.

TECHNOLOGY AND THE STANDARDS

Thirty years later, authors of the *Curriculum and Evaluation Standards for School Mathematics* discussed studying the effects on the graph of changing parameters in the algebraic form (see **fig. 3**). The advantages of the use of technology are highlighted. Spreadsheets and graphing utilities are suggested, which allow students to create accurate graphs quickly so that they can study geometric transformations.

The technology both extends and enhances the activity. Even using circa 1989 graphing calculators, the activity is extended by considering two effects simultaneously. It is enhanced by the ease with which the calculator draws the graphs. Using technology clearly makes the time-consuming, tedious process of graphing more feasible.

Spreadsheet Technology

Figure 4 shows a screen shot from a spreadsheet in which the effects of a are studied. Spreadsheets provide a more numerical approach to this idea. A spreadsheet from twenty years ago would have greatly facilitated the construction of a table of values from which a graph could be drawn by hand. In our experience, producing the table of values slows down the process of graphing by hand. Although spreadsheet graphs allow color-coding and labels to facilitate distinguishing one graph from another, typical graphing calculator tools such as a Trace function were not

Fig. 1 In the 1959 NCTM Yearbook, students were expected to draw graphs and construct tables by hand to compare and contrast different parabolas.

An intermediate algebra class was studying the quadratic function which is determined by $f(x) = ax^2 + bx + c$. They first studied the effects on the graph of the function as changes were made in each one of the parameters a , b , and c while the other two were held constant. The class was also asked to find graphically the real roots, x_1 and x_2 , where they exist when the function is equal to zero. A table of data was then made from the results of different students' graphs (Table 1).

In a discussion following the organization of the first five columns, these four questions were posed:

1. What is the value of $x_1 \cdot x_2$? Record your answer in the table of data.
2. Is there any relation between the product of the roots of $ax^2 + bx + c = 0$ and the arbitrary constants a , b , and c ?
3. What is the value of $x_1 + x_2$? Record your answer in the table.
4. Is there any relation between the sum of the roots of $ax^2 + bx + c = 0$ and the values of a , b , and c ?

(a)

TABLE 1

No.	a	b	c	x_1	x_2	$x_1 \cdot x_2$	$x_1 + x_2$
1	1	-7	12	4	3	12	7
2	1	5	5	-3.4	-1.2	4.1	-4.6
3	1	4	4	-2	-2	4	-4
4	1	4	-5	-5	1	-5	-4
5	2	2	-5	2	.5	1	2.5
6	1	6	-3	.50	-6.5	-3.3	-6
7	3	6	8	.9	-3	-2.7	-2.1
8	1	7	6	-1	-6	6	-7
9	4	0	-1	.5	-.5	-.25	0

After a brief look at the table of data one boy ventured this guess, "The product of the roots is equal to c ." This was called the "Garrett Conjecture," naming it after the boy who had made the guess.

"That seems to be true at least approximately, for 1, 2, 3, 4, 6, and 8, but look at 5, 7, and 9," one of the girls suggested.

The "Garrett Conjecture" was discarded after consideration of this young lady's statement, and the students were encouraged by the teacher to look for likenesses and differences among the two distinct groups of equations stated above. After two other false starts the "Livingston Conjecture" that " $x_1 \cdot x_2 = c/a$ " was proposed. This gained considerable support, although not 100 per cent, apparently because of Examples 6 and 7.

(b)

available. Moreover, constructing the spreadsheet tables involves what may be an extra layer of abstraction for many students. For example, cell D2 in the spreadsheet would be defined by keying $=0.5*A2^2$. When students

want to try a different value for a , they must either create a new column or make a change to the formula in the first cell of the column, then copy it down. Each new investigation would involve several steps.

Fig. 2 Studying the effects of a , b , and c on the graph of $y = ax^2 + bx + c$ required students to use pencil and graph paper to produce many graphs.

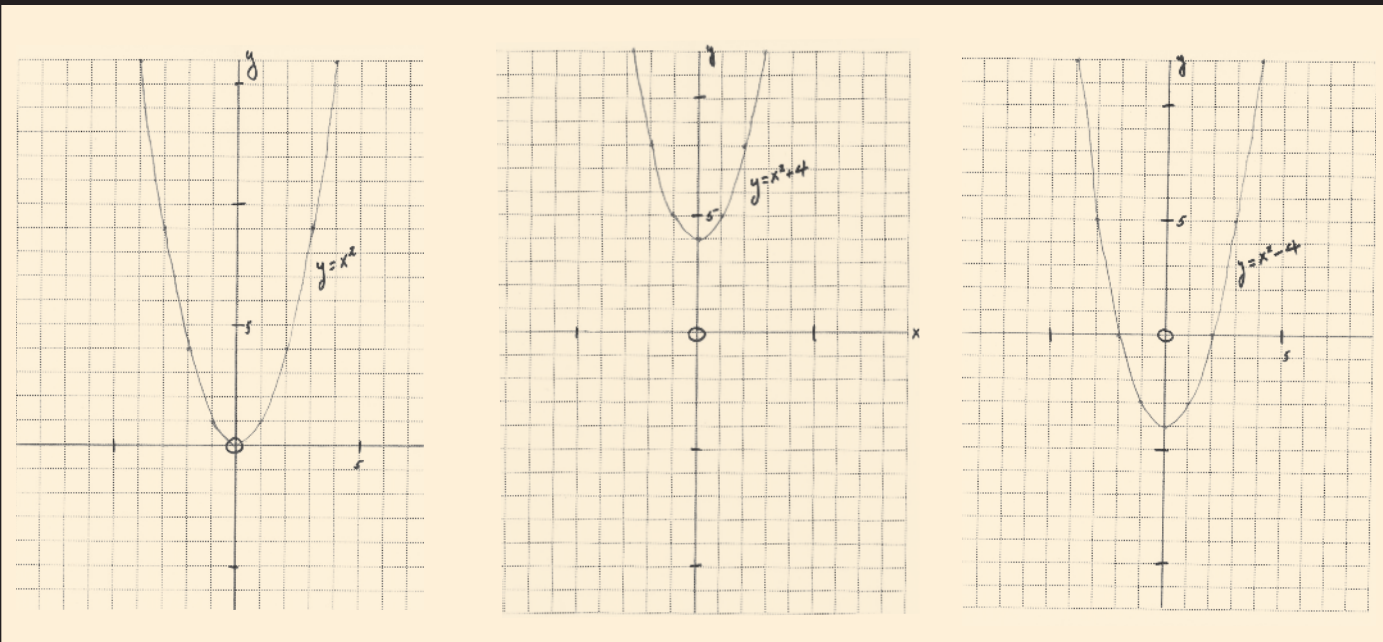
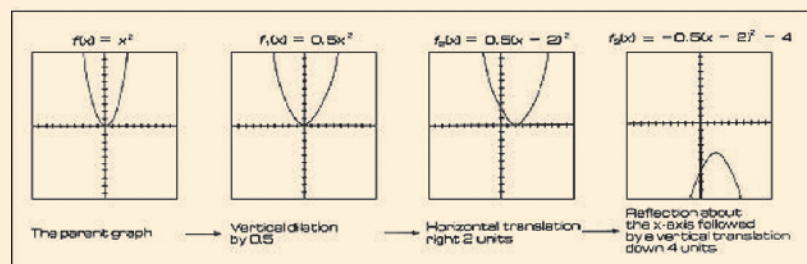


Fig. 3 In the 1989 *Curriculum and Evaluation Standards*, references to spreadsheets and other technology emerge in the expectations for students.

Computing technology provides tools, especially spreadsheets and graphing utilities, that make the study of function concepts and their applications accessible to all students in grades 9-12. This technology makes it possible for students to observe the behavior of many types of functions, including direct and inverse variation, general polynomial, radical, step, exponential, logarithmic, and sinusoidal. All students should use a graphing utility to investigate how the graph of $y = a f(bx + c) + d$ is related to the graph of $y = f(x)$ for various changes of the parameters a , b , c , and d . The effects of these changes can best be expressed by geometric transformations, thereby providing another connection between algebra and geometry. This is illustrated for the function $f(x) = x^2$ by the sequence of graphs in figure 6.3.



In 1991, the *Professional Standards for Teaching Mathematics* was released. The graphing technology was only slightly better than it had been two years earlier. Nevertheless, notice the great improvement over what might have happened in that hypothetical classroom of 1959. At that point, using

the calculator allows all the students to generate and view the graphs up close. See figure 5's teaching vignette. However, notice how busy the graphing window becomes with the addition of so many graphs. Although this may be of no consequence to some students, it might be a hindrance to others.

Graphing-Calculator Technology

The figures that follow demonstrate the use of graphing calculator technology to enrich these activities. **Figure 6**, created using a TI-84 calculator, shows both the power and the limitations of pre-TI-83. To explore the effects of each coefficient on the graph, a series of equations must be entered with two of the coefficients held constant, while the third is varied. Students can compare the graphs to see the results from changes in one of the coefficients. Each student can now consider many graphs simultaneously. However, a limitation of this approach is that the small graphing calculator screen quickly becomes very busy with many graphs. Also, after three or four functions are entered, it is difficult to determine which graph goes with which function.

Figure 7 demonstrates the Transform feature of the TI-83 and TI-84 series calculators. A single general equation ($y = ax^2 + bx + c$) can be entered, and the coefficients dynamically changed, one by one, using the left and right arrow. An advantage of this

Fig. 4 The use of spreadsheets allowed students to quickly generate tables for graphing. In this example, students can study the effects of the leading coefficient.

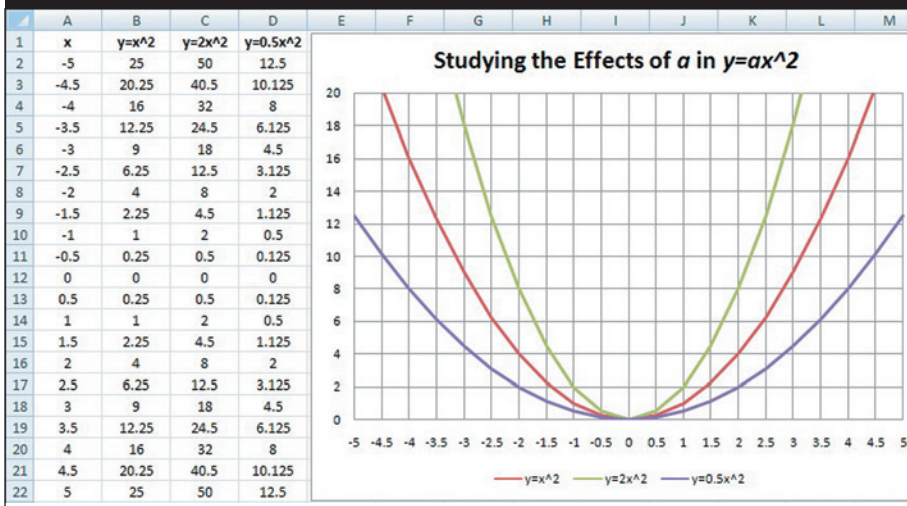
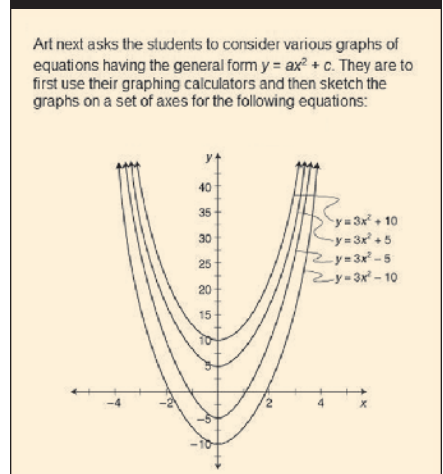


Fig. 5 The 1991 *Professional Teaching Standards* refers to students using graphing technology.



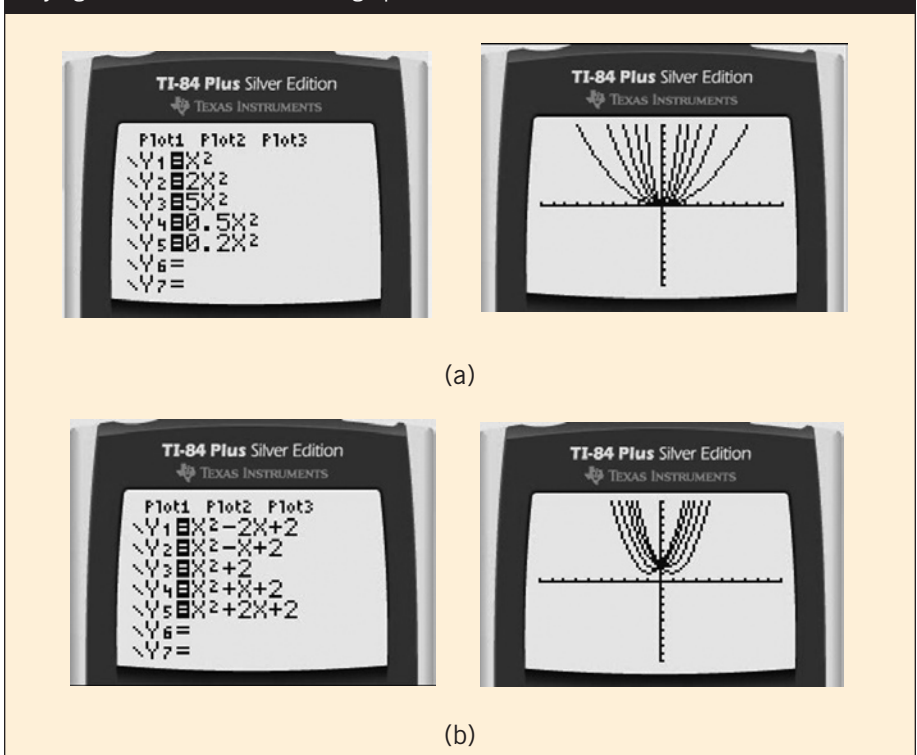
approach is that now only one parabola is graphed on the calculator screen. Manipulating the coefficients appears to cause the parabola to change position or form. However, notice that the graphical and algebraic representations of the function are linked in only one direction. Changes must be made to the algebraic form, and corresponding changes in the graph then observed. Many users often asked themselves, “What if the linkage could be two way?”

In 2000, *Principles and Standards for School Mathematics* was published, which contains this excerpt:

With access to computer algebra systems (CAS)—software on either a computer or calculator that carries out manipulations of symbolic expressions or equations, can compute or approximate values of functions or solutions to equations, and can graph functions and relations—students can easily explore the effects of changes in parameter as a means of better understanding classes of functions. (NCTM 2000, p. 299)

The *Standards* once again visits the idea of using a graphing utility to explore the effects of parameter

Fig. 6 Before the availability of the TI-83 calculator, students had to enter a series of equations with two of the coefficients held constant and a third varied to explore the varying coefficient’s effect on the graph.



changes on the graph of a quadratic function. For the first time, there is an interest in having students observe the effect of changes in the b coefficient. It is also noteworthy that the effect of b is deemed “not as obvious.”

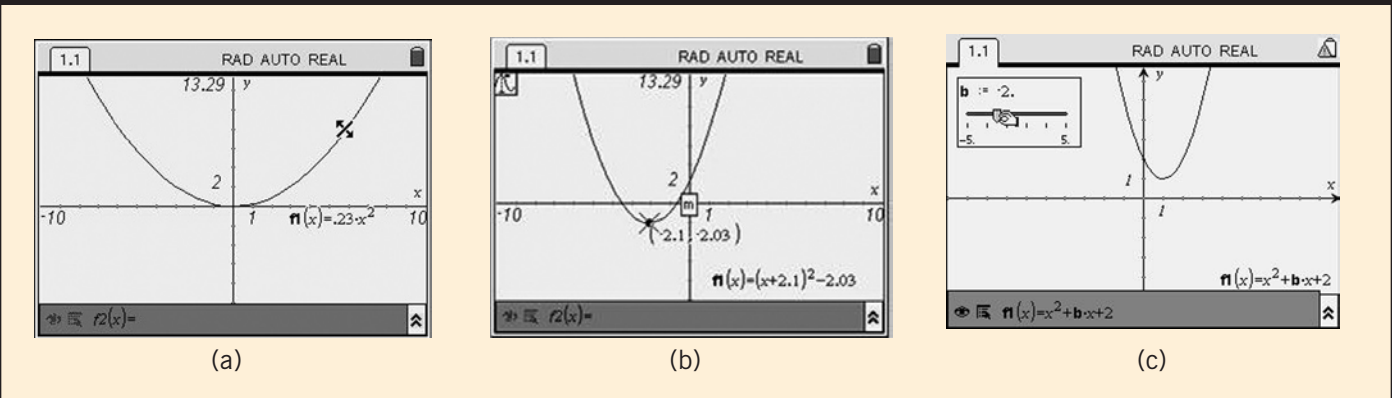
TI-Nspire Technology

The TI-Nspire™ calculator is a second-generation graphing calculator. One significant change is that the linkage between a graph and its algebraic representation is now two way. As with earlier technology, one can

Fig. 7 On the TI-83 and TI-84 calculators, using the Transform feature with quadratic explorations allows students to make changes to the graph and see the results in just one parabola.



Fig. 8 Using the TI-Nspire, there is only one parabola visible on the screen at a time. Moreover, the parabola appears to move in response to movements of the slider that change the value of b (in c).



make changes in the algebraic form and observe the resultant changes in the graph. The new feature is that users can dynamically change the graph by dragging, rotating, or even bending it and observe the resulting changes in the algebraic representation. It is possible to look at the relationship between graph and equation from two perspectives, with each perspective strengthening the other.

The question is, In what ways might such technology positively influence student learning? **Figure 8** demonstrates how the TI-Nspire can be used to explore the effects of variations in the coefficients of a quadratic. The graph of $y = x^2$ is first bent to study the effect of a , then dragged to study the effect of c . Dragging also allows the user to explore the vertex form of a quadratic function. Moreover, using the Trace feature, students

can be guided to observe the relationship between the vertex form and the coordinates of the vertex. Another novel feature of the TI-Nspire is a slider that automatically changes the value of b , but in a way that presents only one graph at a time. As the screen captures in **figure 8** show, at any one time, there is only one parabola visible on the screen. Moreover, the parabola appears to move in response to the slider that changes the value of b . In our experience, the effect of b is not obvious, as the authors of *Principles and Standards* noted (NCTM 2000). Nevertheless, middle school students are quite capable of discerning the effect of the b coefficient when supported by appropriate tools.

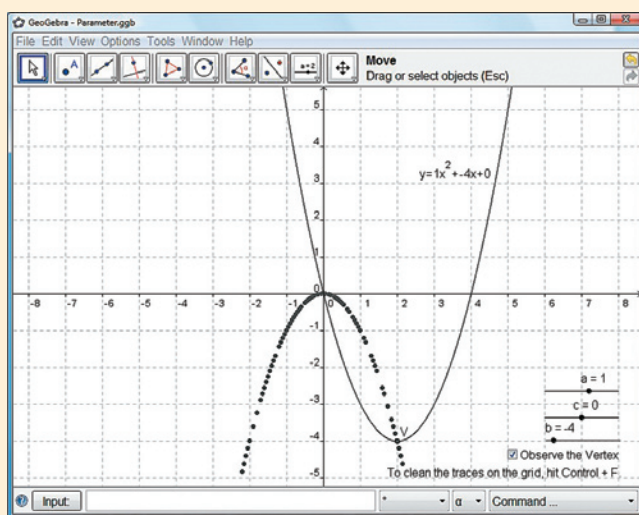
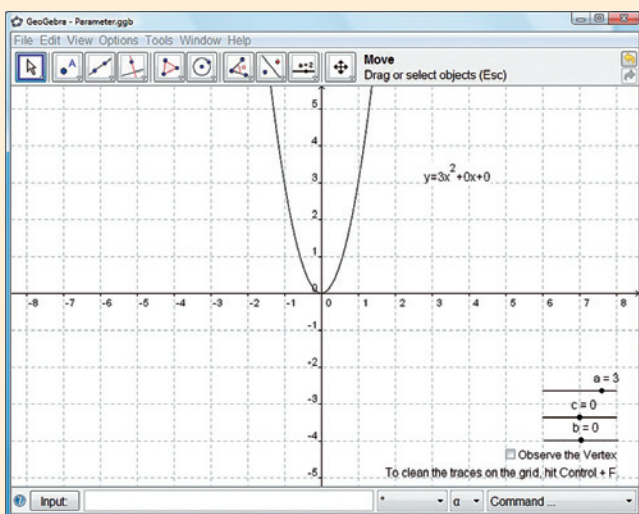
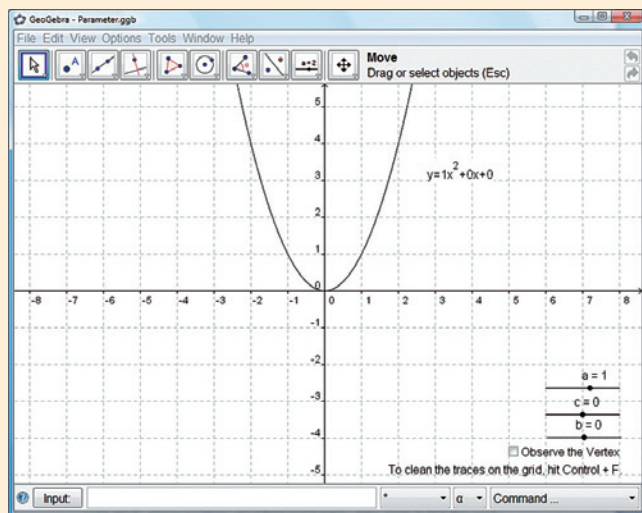
Dynamic Web-Based Software

Teachers can create dynamic worksheets using GeoGebra, free interac-

tive dynamic software, which combines geometry and algebra. Teachers can save GeoGebra worksheets as html files for their students to study from any computer that has an Internet connection. **Figure 9's** screen captures show a dynamic worksheet that we created to study the effects of a , b , and c in $y = ax^2 + bx + c$. Using three sliders for a , b , and c separately enables students to focus on each parameter individually. Moreover, being able to trace the vertex when manipulating the slider for b might help students see the path of the vertex more clearly.

Having easier access to this technology is an advantage. Moreover, seeing the parabola move on the larger computer screen makes it easier to see than on the graphing calculator. GeoGebra enables the user to move a parabola on the screen but not bend it. The relationship between the graph

Fig. 9 Using GeoGebra, students can use three sliders for a , b , and c in the graph of $y = ax^2 + bx + c$ separately, which enables them to focus on each parameter separately.



and the algebraic form is neither one way nor two way, but about halfway linked.

Changes can be made to the algebraic form, and the user can observe the changes to the graphical representation. The link from the algebraic form to the graphical form is there. Although the user can make changes to the graph and watch the algebraic form, it can be moved only, not bent. The link from the graphic form to the algebraic form is only partial.

CLASSROOM VIDEOS

We made two short videos of students using the TI-Nspire to explore quadratic functions. Both videos were captured in a class of seventh-graders studying algebra 1. There were only twelve students in the class, and their teacher regularly used a class set of TI-84 calculators during instruction.

The Effect of the Coefficient a

The students used the TI-Nspire to study the effect of the coefficient a on the graph of a quadratic function, similar to the activity demonstrated earlier in the TI-Nspire movie. As we reviewed this video, we noticed right away that the students were intrigued by the ability to bend the parabola. We think this video also showed the ease with which students were able to answer questions about their manipulation of the graph. In fact, they easily connected narrowing the parabola with larger values of a and widening the parabola with smaller values of a . They also easily connected flipping the parabola upside down with negative values of a .

Finally, we were pleased with the fun that students seemed to have with a “match the parabola” game, as well as what they seemed to learn from it. In the game, a graph was drawn by hand on the whiteboard where the TI-Nspire screen was being projected. Students were invited

to change the equation on the demonstration calculator to match the hand-drawn graph. As they played the game, they were able to use the knowledge that they had conjectured by linking the graph to the equation to make decisions about the link connecting the equation to the graph.

The Effect of the Coefficient b

In the second video, the students used the TI-Nspire to study the effect of the coefficient b on the graph of a quadratic function by manipulating a slider to change the value of b . We were hoping that the students would be able to go beyond noticing that the parabola appears to move both horizontally and vertically to noticing that the vertex of the given parabola is itself moving along a parabolic path. To help them see this, we had an elaborate extension activity planned that uses the spreadsheet capability of the TI-Nspire, but something happened early in the initial whole-class discussion of the slider activity that took that discussion further than planned.

We were fascinated by one student's observation that the parabola was moving on "an invisible, upside-down parabola." Even more intriguing

was the ease with which the rest of the class seemed to embrace his idea. Finally, when another student suggested replacing the first student's dot-graph depiction of what he was seeing with a continuously drawn parabola, and then observed that the vertex of the moving parabola was moving along the upside-down parabola, these middle school students had made an observation that most of our graduate students had failed to make when using the older technology.

DISCUSSION

We think that this historical perspective on investigating the relationships between the coefficients of a quadratic function and its graph has demonstrated that the power of technology can be exploited to enrich students' mathematical explorations. Although the use of technology has not altered the *content* of these activities, when used appropriately, technology holds promise for altering the *process* of the activities.

From 1959 to the present, tasks such as conjecturing, discovering, and problem solving have been valued in mathematics classrooms. As has often been postulated, because technology can handle enormous computational loads quickly and easily, students' minds are freed to consider the larger conceptual issues and higher-level-thinking tasks underlying the mathematical activity.

As we have traced the progression of technologies applied to these activities, it is clear that it is continually evolving.

When one reflects

that graphing calculators did not exist twenty-five years ago, one can hardly imagine the capabilities of handheld technologies twenty-five years into the future. Just as it does today, technology will continue to make things that are difficult easier, and things that are infeasible possible.

REFERENCES

- National Council of Teachers of Mathematics (NCTM). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: NCTM, 1989.
- . *Professional Standards for Teaching Mathematics*. Reston, VA: NCTM, 1991.
- . *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.
- Smith, Eugene P., and Kenneth B. Henderson. "Proof: Some Common Uses of the Term Proof." In *The Growth of Mathematical Ideas, Grades K-12*, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics (NCTM), pp. 111–81. Washington, DC: NCTM, 1959.

View this article on the NCTM Web site at www.nctm.org/mtms to see videos of students using technology that was discussed in this article.



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invention. **S. Aslı Özgün-Koca**, aokoca@wayne.edu, teaches mathematics and secondary mathematics education courses at Wayne State University. She is interested in the use of technology in mathematics instruction.

The Casio fx-7500G was a state-of-the-art graphing calculator in 1989.

